MAC 2312 – Calculus II

Section 11.10 – Taylor & Maclaurin Series

Taylor Series

If f is a function and $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

for all x in the open interval containing c, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Maclaurin Series

A Taylor series centered at $c=0$

If f is a function and $f(x) = \sum a_n x^n$ for all x in the open interval $(-r, r)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Theorem

If a function f has derivatives of all orders throughout an interval containing c, and if

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \left(R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \quad c < z < x \right)$$

for every x in the interval, then $f(x)$ is represented by a Taylor series at $x=c$.

Theorem For every real number x

$$\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0$$

Conclusion: $n!$ increases faster than x^n

Example 1 A Maclaurin Series

Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for every real number x.

$f(x) = \sin x$ $\quad$ $f(0) = \sin(0) = 0$
$f'(x) = \cos x$ $\quad$ $f'(0) = \cos(0) = 1$
$f''(x) = -\sin x$ $\quad$ $f''(0) = -\sin(0) = 0$
$f'''(x) = -\cos x$ $\quad$ $f'''(0) = -\cos(0) = -1$
$f^{(4)}(x) = \sin x$ $\quad$ $f^{(4)}(0) = \sin(0) = 0$

the form for a maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \cdots$$

$$f(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)x^3}{6} + \frac{0 \cdot x^4}{24} + \cdots$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

the difficult part is now to rewrite the series using summations. Look for patterns.

$$\boxed{f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

alternating

$\boxed{\text{The proof part}}$

prove that it represents $\sin x$ for every real number x

Since n is a positive integer, we know that one of the following is true

$$|f^{(n+1)}(x)| = |\cos x| \quad \text{or} \quad |f^{(n+1)}(x)| = |\sin x|$$

so

$$|f^{(n+1)}(z)| \leq 1 \text{ for every number } z$$

let $c=0$ (maclaurin series)

$$|R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}\right| |x|^{n+1} \leq \underbrace{\frac{|x|^{n+1}}{(n+1)!}}$$

a remainder term of a polynomial

recall $\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0$

So it follows by the squeeze theorem that $\lim_{n\to\infty} |R_n(x)| = 0$ and the maclaurin series representation of $\sin x$ is true for all real values of x.

Example 2 Find a Maclaurin series for $f(x) = e^x$ for every real number x.

$$f(x) = e^x \qquad f(0) = e^0 = 1$$
$$f'(x) = e^x \qquad f'(0) = e^0 = 1$$
$$f''(x) = e^x \qquad f''(0) = e^0 = 1$$
$$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Example 3

Find the Maclaurin series for $x^2 \sin x$

The long way

$f(x) = x^2 \sin x$ $\qquad f(0) = 0$

$f'(x) = 2x\sin x + x^2 \cos x$ $\qquad f'(0) = 0$

$f''(x) = 2\sin x + 4x\cos x - x^2 \sin x$ $\qquad f''(0) = 0$

$f'''(x) = 6\cos x - 6x\sin x - x^2\cos x$ $\qquad f'''(0) = 6$

$f^{(4)}(x) = -12\sin x - 8x\cos x - x^2 \sin x$ $\qquad f^{(4)}(0) = 0$

$f^{(5)}(x) =$ $\qquad f^{(5)}(0) =$

$f^{(6)}(x) =$ $\qquad f^{(6)}(0) = 0$

$$f(x) = \cancel{f(0)}^{0} + \cancel{f'(0)}^{0}x + \cancel{f''(0)}^{0}\frac{x^2}{2!} + \overset{6}{f'''(0)}\frac{x^3}{3!} + \cdots$$

$$f(x) = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \cdots$$

Alternate method (much more efficient)

we previous found the maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\text{so } x^2 \sin x = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots + (-1)^n \frac{x^{2n+1} \cdot x^2}{(2n+1)!}$$

$$\text{so } \boxed{x^2 \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!}}$$

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$$x^2 \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$$

$\boxed{\text{now for the radius of convergence for } x^2 \sin x}$

use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2(n+1)+3}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n \cdot x^{2n+3}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2n+5}}{x^{2n+3}} \cdot \frac{(2n+1)!}{(2n+3)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{x^{2n}}\, x^5}{\cancel{x^{2n}}\, x^3} \cdot \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)} \cdot x^2\right| = 0$$

So by ratio test, this series converges for all real values of x.

(*) I will not ask you find / prove that the maclaurin series converges for all values of x that are real on the exam.

Example 4

Find the Taylor Series for $\sin x$ at $\frac{\pi}{6}$

$f(x) = \sin x$ $\quad f(\frac{\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$

$f'(x) = \cos x$ $\quad f'(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$

$f''(x) = -\sin x$ $\quad f''(\pi/6) = -\sin(\pi/6) = -1/2$

$f'''(x) = -\cos x$ $\quad f'''(\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(\pi/6) = \sin(\pi/6) = 1/2$

now we start looking for patterns

the format for Taylor Polynomials

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + \cdots$$

$$\sin x = f(\pi/6) + f'(\tfrac{\pi}{6})(x-\pi/6) + f''(\tfrac{\pi}{6})\frac{(x-\pi/6)^2}{2!} + \cdots$$

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\frac{\pi}{6}) - \frac{1}{2}\frac{(x-\pi/6)^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\pi/6)^3}{3!} + \frac{1}{2}\frac{(x-\pi/6)^4}{4!} + \cdots$$

we will need to split this into two series and essentially create a piecewise function.

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$\boxed{\text{function 1}}$ n is even (+ zero)

$$\frac{1}{2} - \frac{1}{2}\,\frac{(x-\pi/6)^2}{2!} + \frac{1}{2}\,\frac{(x-\pi/6)^4}{4!} - \frac{1}{2}\,\frac{(x-\pi/6)^6}{6!} + \cdots$$

$$\frac{1}{2}\left[(-1)^{n/2}\,\frac{1}{n!}\left(x-\frac{\pi}{6}\right)^n\right] \quad n = 0, 2, 4, \ldots$$

$\boxed{\text{function 2}}$ n is odd

$$\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}\,\frac{\left(x-\frac{\pi}{6}\right)^3}{3!} + \frac{\sqrt{3}}{2}\,\frac{\left(x-\frac{\pi}{6}\right)^5}{5!} + \cdots$$

$$\frac{\sqrt{3}}{2}\left[(-1)^{\left(\frac{n-1}{2}\right)}\,\frac{1}{n!}\left(x-\frac{\pi}{6}\right)^n\right] \quad n = 1, 3, 5$$

$$\boxed{\sin x = \begin{cases} \frac{1}{2}\left[(-1)^{n/2}\cdot\frac{1}{n!}\,(x-\pi/6)^n\right] & n = 0, 2, 4, \ldots \\ \frac{\sqrt{3}}{2}\left[(-1)^{\frac{n-1}{2}}\cdot\frac{1}{n!}\left(x-\frac{\pi}{6}\right)^n\right] & n = 1, 3, 5, \ldots \end{cases}}$$

Example 5 Find the Taylor series for 10^x when $c=0$

Note: This is actually a Maclaurin series

recall if $f(x) = a^x$
$f'(x) = a^x \cdot \ln a$

note that $a=10$

$$\begin{aligned}
f(x) &= a^x & f(0) &= a^0 = 1 \\
f'(x) &= a^x \ln a & f'(0) &= a^0 \ln a = \ln a \\
f''(x) &= a^x (\ln a)^2 & f''(0) &= a^0 (\ln a)^2 = (\ln a)^2 \\
f'''(x) &= a^x (\ln a)^3 & f'''(0) &= a^0 (\ln a)^3 = (\ln a)^3
\end{aligned}$$

format

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

$$10^x = 1 + (\ln 10)x + (\ln 10)^2 \frac{x^2}{2!} + (\ln 10)^3 \frac{x^3}{3!} + \cdots$$

$$\boxed{10^x = \sum_{n=0}^{\infty} (\ln 10)^n \frac{x^n}{n!}}$$

final answer (cleaned up)

$$\boxed{\sin^{-1}(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right) + \frac{2}{3\sqrt{3}}\left(x-\frac{1}{2}\right)^2 + \frac{8}{9\sqrt{3}}\left(x-\frac{1}{2}\right)^3}$$

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Example 6

Find the first four terms of the Taylor Series for $f(x) = \sin^{-1}(x)$ when $c = \frac{1}{2}$

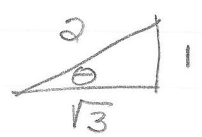
$f(x) = \sin^{-1}(x)$ $\qquad$ $f(\frac{1}{2}) = \pi/6$

$f'(x) = \dfrac{1}{(1-x^2)^{1/2}}$ $\qquad$ $f'(\frac{1}{2}) = \dfrac{2}{\sqrt{3}}$

$f''(x) = \dfrac{x}{(1-x^2)^{3/2}}$ $\qquad$ $f''(\frac{1}{2}) = \dfrac{4}{3\sqrt{3}}$

$f'''(x) = \dfrac{(2x^2+1)}{(1-x^2)^{5/2}}$ $\qquad$ $f'''(\frac{1}{2}) = \dfrac{16}{3\sqrt{3}}$

Recall: To find $\sin^{-1}x$, set up a triangle



or think about the sin of what angle is $\frac{1}{2}$?

the Taylor Polynomial (Series)

$$\sin^{-1}(x) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(x-\tfrac{1}{2}) + f''(\tfrac{1}{2})\frac{(x-\frac{1}{2})^2}{2!} + f'''(\tfrac{1}{2})(x-\pi/6)^3 \cdot \frac{1}{3!}$$

$$\sin^{-1}(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right) + \frac{\cancel{4}^{\,2}}{3\sqrt{3}} \cdot \frac{1}{\cancel{2}}\left(x-\frac{1}{2}\right)^2 + \frac{\cancel{16}^{\,8}}{3\sqrt{3}} \cdot \frac{1}{\cancel{6}_{\,3}}\left(x-\frac{1}{2}\right)^3$$

Example 7

Approximate $\int_0^1 \sin(x^2)\,dx$ to four decimal places

Let's use a maclaurin series representation

Recall that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

so $\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

$$\int_0^1 \sin(x^2)\,dx = \frac{x^3}{3} - \frac{x^7}{7\cdot 6} + \frac{x^{11}}{11\cdot 120} - \frac{x^{15}}{15\cdot 7}\Bigg|_0^1$$

$$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \cdots$$

$$\cong \boxed{0.3103}$$

Section 11.10 | Practice Problems

Here are some problems for you to try on your own.

Find the maclaurin series for each of the following functions:

1. $f(x) = \cos x$

2. $f(x) = e^{2x}$

3. $f(x) = x \sin 3x$

4. $f(x) = x^2 e^x$

Find the Taylor series for each of the following functions:

5. $f(x) = \cos x$ at $c = \pi/3$

6. $f(x) = \dfrac{1}{x}$ at $c = 2$

7. Find the first four terms of the Taylor Series for $f(x) = xe^x$ when $c = -1$

8. Use a maclaurin series to approximate the integral accurate to 4 decimal places

$$\int_0^{0.5} \cos(x^2)\,dx$$

Answers and helpful hints for 11.9 practice problems.

(1) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

(3) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1} x^{2n+2}}{(2n+1)!}$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$

(5) $\cos x = \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} \cdot \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2$

$+ \frac{1}{2} \cdot \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^3$

sign pattern: + − − + + − − +

Convert it using $\cos(x) = \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right)$

use

$\cos(\alpha + \beta) =$

$$\boxed{\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(x - \frac{\pi}{3}\right)^{2n} \cdot \frac{1}{(2n)!} + \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(x - \frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}}$$

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(6) $f^n(x) = (-1)^n \dfrac{n!}{x^{n+1}}$ $\quad f^n(2) = (-1)^n \dfrac{n!}{2^{n+1}}$

$f(x) = 1/x$ $\quad f(2) = \frac{1}{2}$

$f'(x) = -1/x^2$ $\quad f'(2) = \dfrac{-1}{(2)^2}$

$f''(x) = 2/x^3$ $\quad f''(2) = \dfrac{2}{(2)^3}$

$f'''(x) = -6/x^4$ $\quad f'''(2) = \dfrac{-6}{(2)^4}$

final answer

$$\boxed{\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2^{n+1}}\right](x-2)^n}$$

(7) $xe^x = -e^{-1} + \left(\frac{e^{-1}}{2!}\right)(x+1)^2 + \left(\frac{2e^{-1}}{3!}\right)(x+1)^3$
$+ \left(\frac{3e^{-1}}{4!}\right)(x+1)^4$

(8) $(0.5) - \dfrac{(0.5)^5}{10} + \dfrac{(0.5)^9}{9(24)} - \dfrac{(0.5)^{17}}{17(720)} +$

$0.5 - 0.00312 + \underbrace{0.0000091}$ so stop at 3 terms

$\boxed{0.4969}$